Matrix Inverse

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Stimulus-response computations

- Many of the linear systems of equations arising in engineering and science are derived from conservation laws.

$$[A]\{x\} = \{b\}$$

forcing functions or external stimuli

$$[\text{Interactions}]\{\text{response}\} = \{\text{stimuli}\}$$

System's state or response

Matrix inverse

- If a matrix [*A*] is square

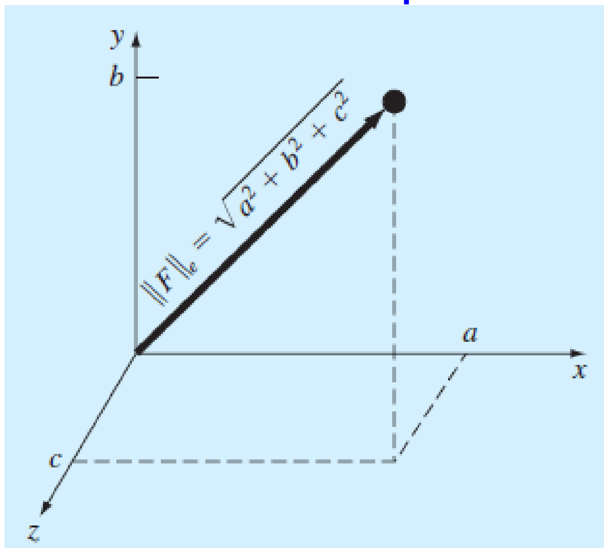
  $[A][A]^{-1}=[A]^{-1}[A]=[I]$

Vector norms

- *p*-norm

$$\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

Euclidean space



$p = 1:$ sum of the absolute values $\quad \|X\|_1 = \sum_{i=1}^{n} |x_i|$

$p = 2:$ Euclidian norm (length) $\quad \|X\|_2 = \|X\|_e = \sqrt{\sum_{i=1}^{n} x_i^2}$

$p = \infty:$ maximum $-$ magnitude $\quad \|X\|_\infty = \max_{1 \le i \le n} |x_i|$

Norm of vector in MATLAB

- norm(*X*,*p*)

p	Vector
1	sum(abs(X))
2	sum(abs(X).^2)^(1/2)
Positive, real-valued numeric p	sum(abs(X).^p)^(1/p)
Inf	max(abs(X))

Matrix norms

column - sum norm $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$

Frobenius norm $\|A\|_f = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2}$

row - sum norm $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$

spectral norm (2 norm) $\|A\|_2 = (\mu_{\max})^{1/2}$

μ_{max} is the largest eigenvalue of $[A]^T[A]$

Norm of matrix in MATLAB

- norm(*A*,*p*)

p	**Matrix**
1	max(sum(abs(A)))
2	max(svd(A))
inf	max(sum(abs(A')))
‘fro’	sqrt(sum(diag(A'*A)))

Matrix condition number

- $\mathrm{Cond}[A]=||A||\cdot||A^{-1}||$
  This number greater than or equal to 1

Relative error of the norm of the computed solution

$t-c$ digits

$$\frac{\|\Delta X\|}{\|X\|} \le \mathrm{Cond}[A]\frac{\|\Delta A\|}{\|A\|}$$

$Cond[A] = 10^c$

Relative error of the norm of the coefficients of [A]

t digit precision
(rounding errors, the order of 10^{-t})

Condition number of matrix in MATLAB

- cond(A,p)

  = norm(A,p) * norm(inv(A),p)

If p is...	Then cond(A,p) returns the...
1	1-norm condition number
2	2-norm condition number
'fro'	Frobenius norm condition number
inf	Infinity norm condition number

Matrix condition evaluation of the Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

$$[A] = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \qquad [A]^{-1} = \begin{bmatrix} 9 & -18 & 10 \\ -36 & 96 & -60 \\ 30 & -90 & 60 \end{bmatrix}$$

norm(A,inf) = 1.8333 norm(inv(A),inf) = 408.0000

cond(A,inf) = 1.833 * 408.0000 = 748.0000

- This system is ill-conditioned because its condition number is much greater than 1.
- *c* = log10(748) = 2.87. Hence, the last 3 significant digits of the solution could exhibit rounding errors.

Exercise: Condition number

- Compute the condition number based on row-sum norm with/without normalization in each row by hand and by Matlab
- Compute the significant digits of the solution x of Ax=b that exhibit rounding errors.

$$\begin{bmatrix} 1 & 4 & 9 & 16 & 25 \\ 4 & 9 & 16 & 25 & 36 \\ 9 & 16 & 25 & 36 & 49 \\ 16 & 25 & 36 & 49 & 64 \\ 25 & 36 & 49 & 64 & 81 \end{bmatrix}$$

Reference

- Steven C. Chapra "Applied Numerical Methods with MATLAB", 3rd ed., McGraw Hill, 2012.