Numerical Methods

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Mathematical modeling, numerical methods, and problem solving

- Mathematical models
  - Scientific principles to simulate behavior of simple physical system
- Numerical methods
  - Means to generalize solutions

Simple mathematical model

- Model function

$$\begin{matrix}\text{Dependent}\\\text{variable}\end{matrix} = f\left(\begin{matrix}\text{independent}\\\text{variables}\end{matrix},\ \text{parameters},\ \begin{matrix}\text{forcing}\\\text{functions}\end{matrix}\right)$$

- **Dependent variable** - a characteristic that reflects the behavior or state of the system
- **Independent variables** - such as time and space, along which system's behavior is being determined
- **Parameters** - constants reflective of the system's properties or composition
- **Forcing functions** - external influences acting upon the system

Model for bungee jumper

- Explicit model function

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$

- Dependent variable - velocity v
- Independent variables - time t
- Parameters - mass m, drag coefficient c_d
- Forcing function - gravitational acceleration g

Model for bungee jumper (cont.)

- Newton's 2nd law

$$a = \frac{F}{m}$$

$$F = F_D + F_U = mg - c_d v^2$$

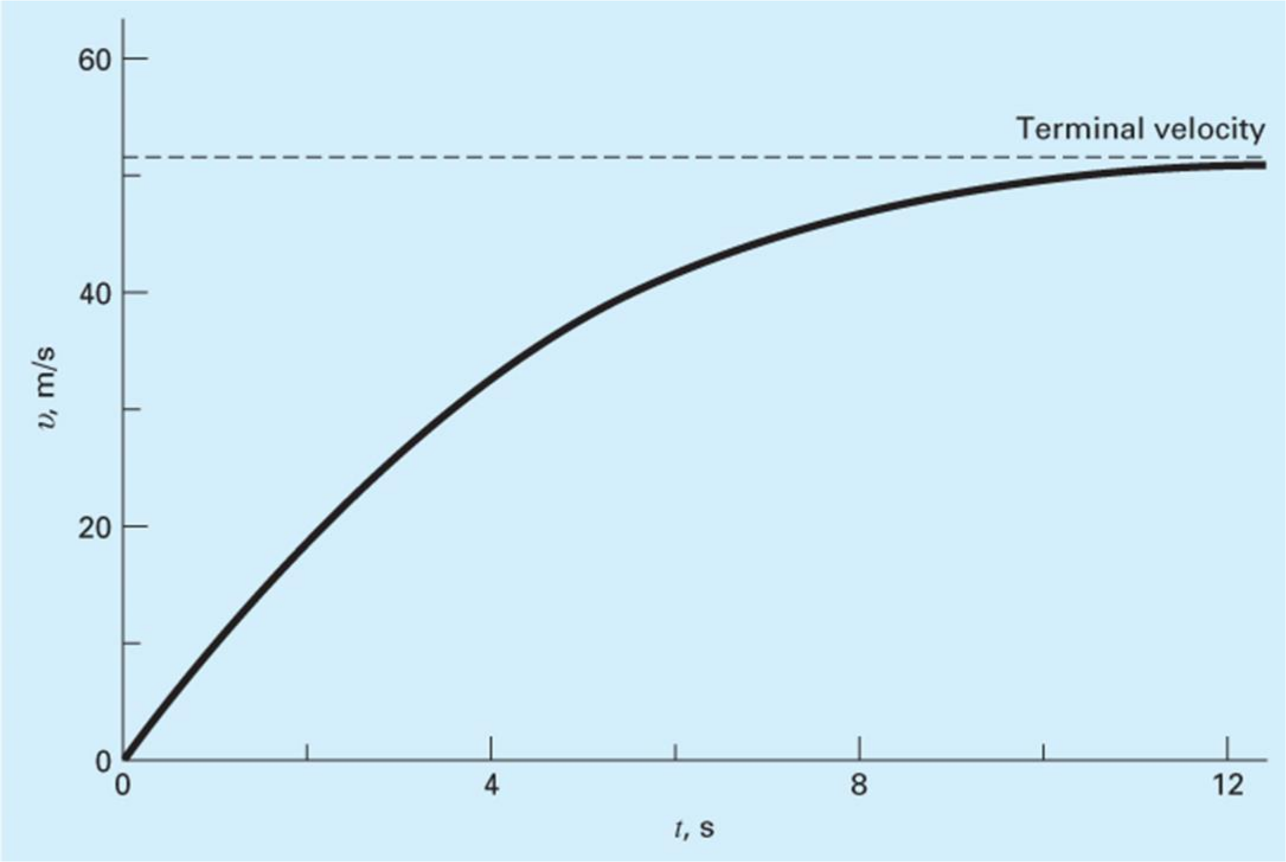
$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Closed-form solution for bungee jumper problem

- Example: a 68.1 kg jumper, assuming a drag coefficient of 0.25 kg/m. Determine velocity from t=0:1:12.



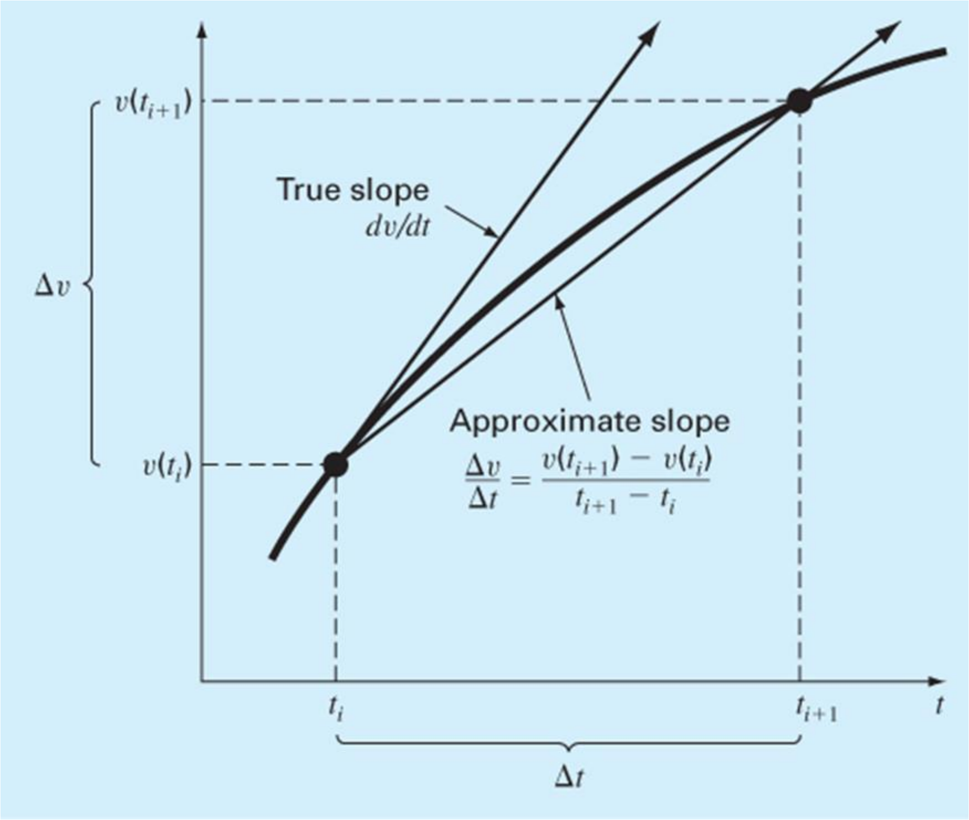
MATLAB simulation
(Please practice Chap 2 MATLAB Fundamental)

```
% Assignment
g=9.81;
m=68.1;
cd=0.25;
% Create an array or a vector
t1=0:1:12;
% Use of built-in function
v1=sqrt(g*m/cd)*tanh(sqrt(g*cd/m)*t1);
% Graphics
plot(t1,v1)
```

Numerical method for bungee jumper problem

- Some system models are given as implicit functions

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$



$$\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

Euler's method

- Substituting the finite difference into the differential equation

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

$$\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c_d}{m} v^2$$

- Solve for

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m} v(t_i)^2 \right] (t_{i+1} - t_i)$$

new = old + slope × step

MATLAB simulation

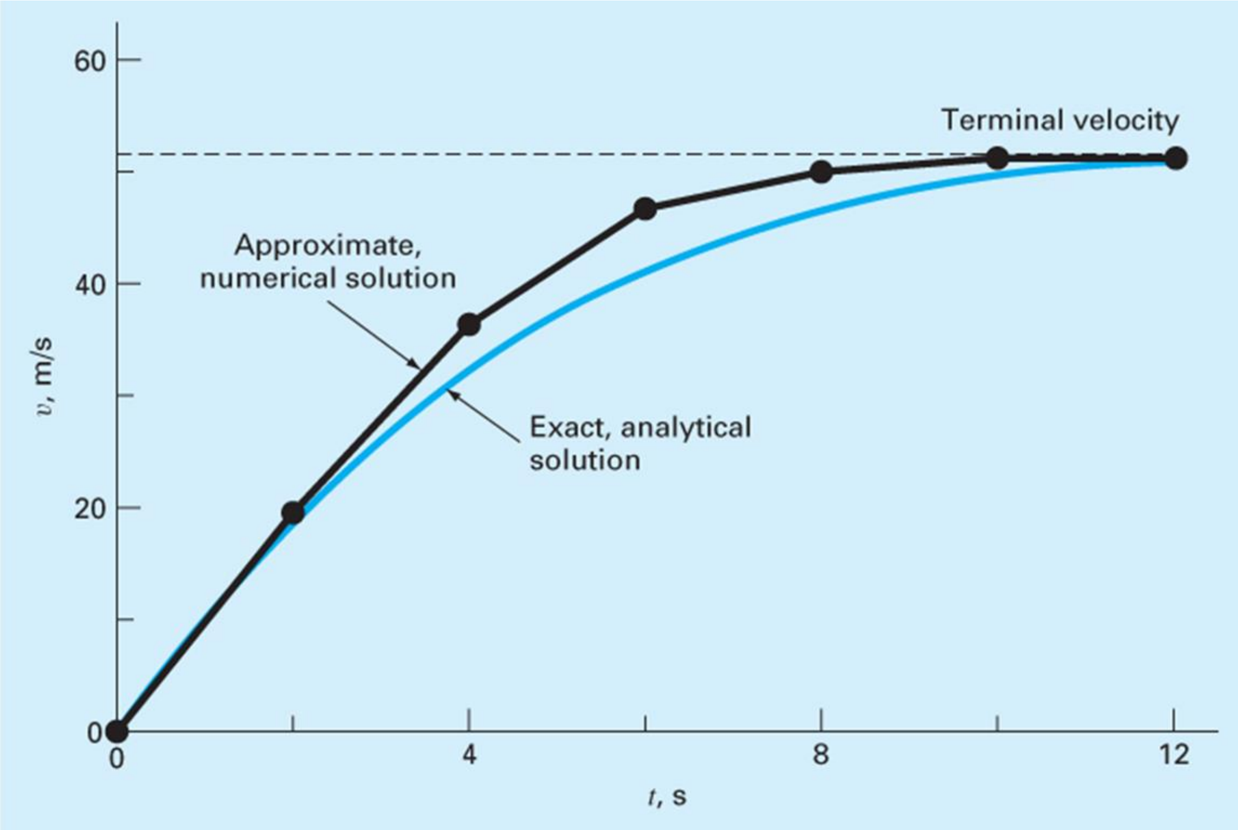
```
g=9.81;
m=68.1;
cd=0.25;

t2=0:2:12;
v2(1)=0;
for i=1:1:length(t2)-1
    v2(i+1)=v2(i)+(g-(cd/m)*v2(i)^2)*(t2(i+1)-t2(i));
end

plot(t2,v2)
```

Numerical results

- Initial value: $v(0)$



- How do we improve the solution?

MATLAB simulation

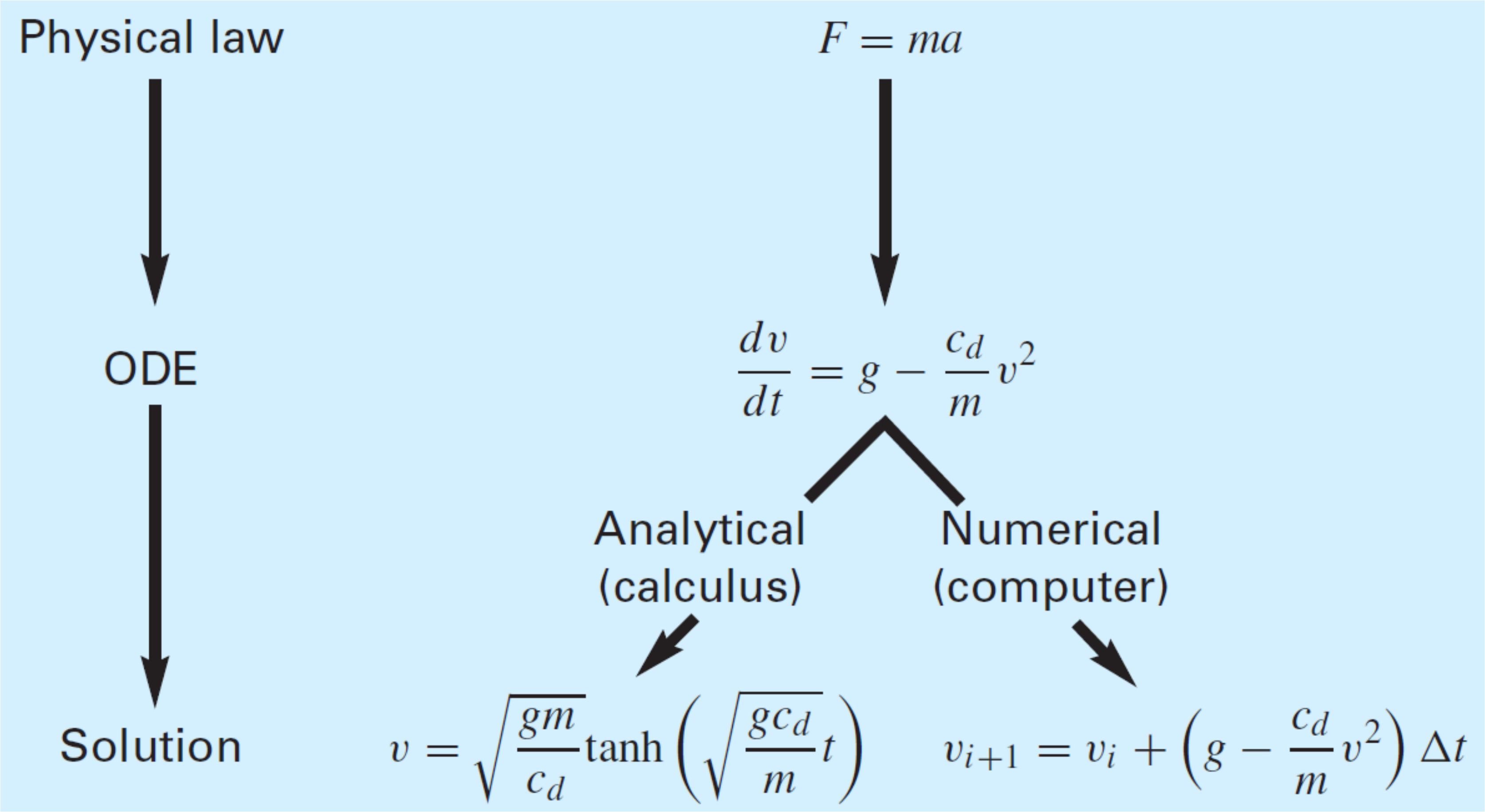
```
g=9.81;
m=68.1;
cd=0.25;

t1=0:1:12;
v1=sqrt(g*m/cd)*tanh(sqrt(g*cd/m)*t1);

t2=0:2:12;
v2(1)=0;
for i=1:1:length(t2)-1
    v2(i+1)=v2(i)+(g-(cd/m)*v2(i)^2)*(t2(i+1)-t2(i));
end

plot(t1,v1,'b',t2,v2,'k.-');
xlabel('t, s');
ylabel('v, m/s');
```

Analytic and numerical methods for bungee jumper



Analytic method (from Edward Yang, University of Toronto)

You are given the following differential equation with the initial condition, $v(t = 0) = 0$

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

Multiply both sides by m/c_d

$$\frac{m}{c_d}\frac{dv}{dt} = \frac{m}{c_d} g - v^2$$

Define $a = \sqrt{mg / c_d}$

$$\frac{m}{c_d}\frac{dv}{dt} = a^2 - v^2$$

$$\frac{1}{a^2 - v^2}\frac{dv}{dt} = \frac{c_d}{m}$$

This is a first order differential equation of the form: $P(y)\frac{dy}{dx} = Q(x)$

https://www.studeersnel.nl/nl/document/university-of-toronto/engineering-mathematics-i/college-aantekeningen/6-lecture8-rootfinding-secant-method/326571/view?has_flashcards=1

$$\frac{m}{c_d}\frac{dv}{dt} = a^2 - v^2$$

Integrate by separation of variables,

$$\int \frac{dv}{a^2 - v^2} = \int \frac{c_d}{m} dt$$

A table of integrals can be consulted to find that

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

Therefore, the integration yields

$$\frac{1}{a}\tanh^{-1}\frac{v}{a} = \frac{c_d}{m}t + C$$

If $v = 0$ at $t = 0$, then because $\tanh^{-1}(0) = 0$, the constant of integration $C = 0$ and the solution is

$$\frac{1}{a}\tanh^{-1}\frac{v}{a} = \frac{c_d}{m}t$$

This result can then be rearranged to yield

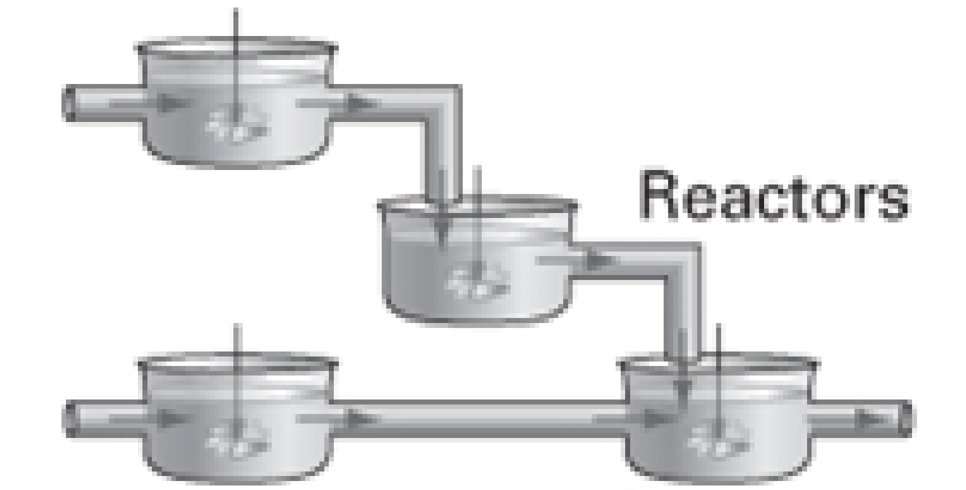
$$v = \sqrt{\frac{gm}{c_d}}\tanh\left(\sqrt{\frac{gc_d}{m}}t\right)$$

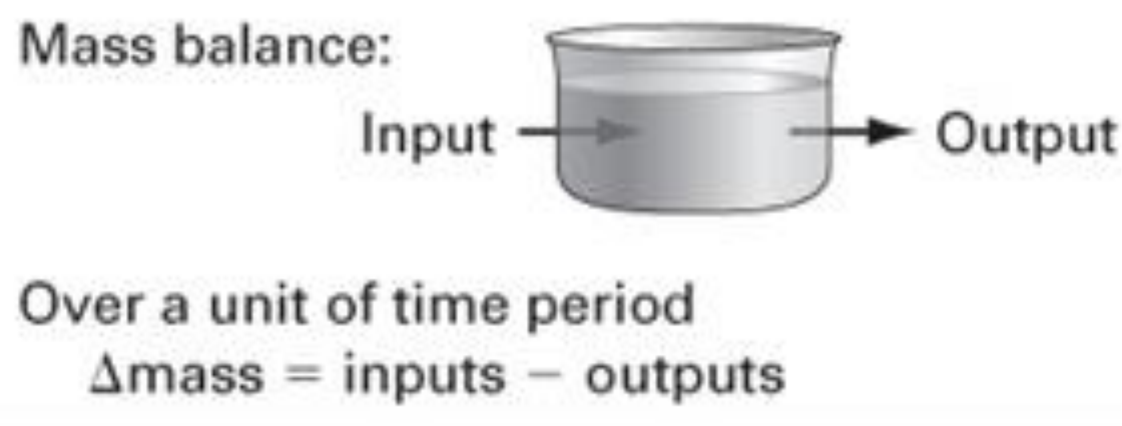
Recall:
We define $\tanh^{-1}(x) = y$ whenever $\tanh(y) = x$

Conservation laws

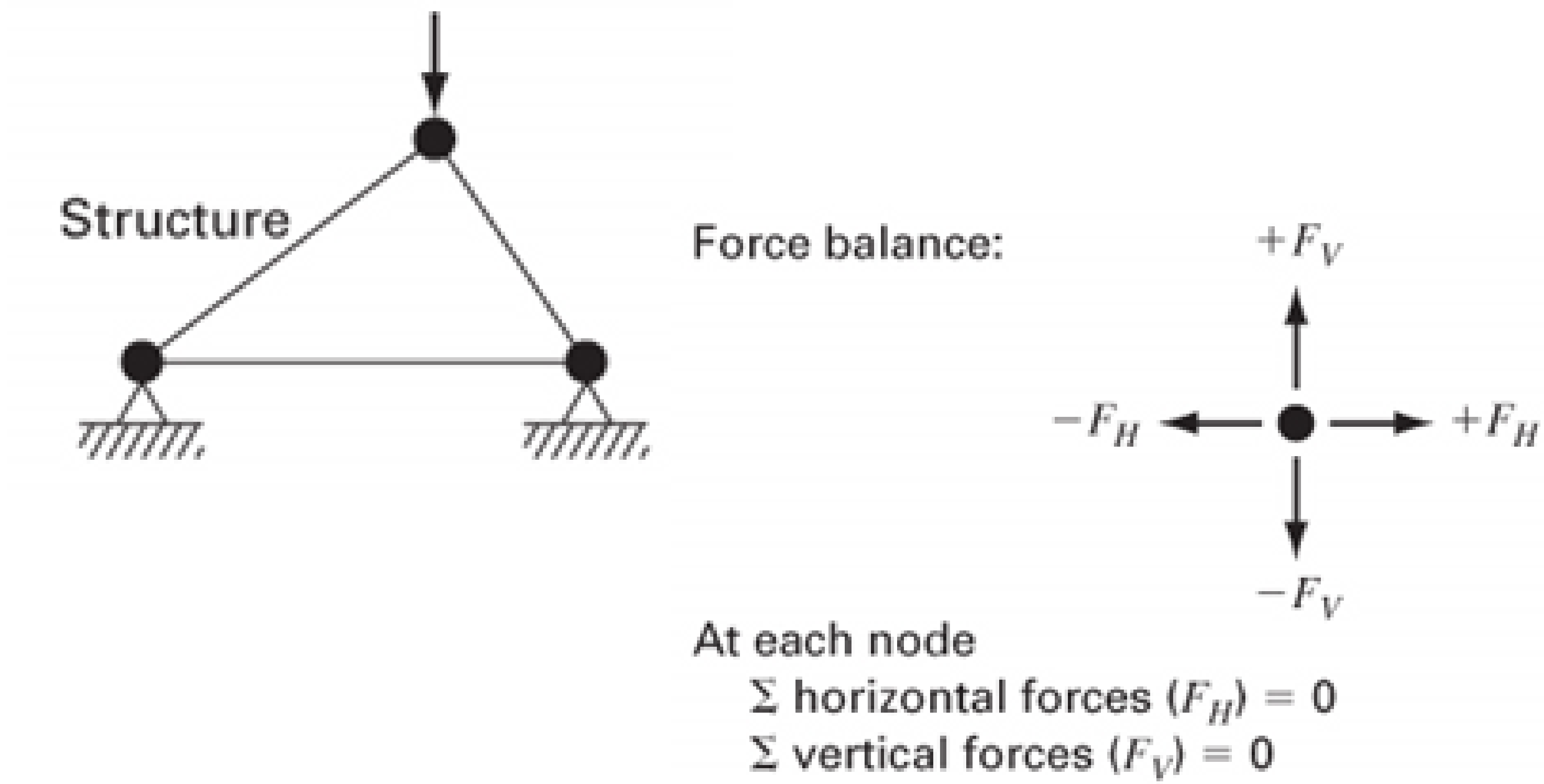
- Conservation of mass
- Conservation of momentum
- Conservation of charge
- Conservation of energy

Conservation of mass in chemical engineering



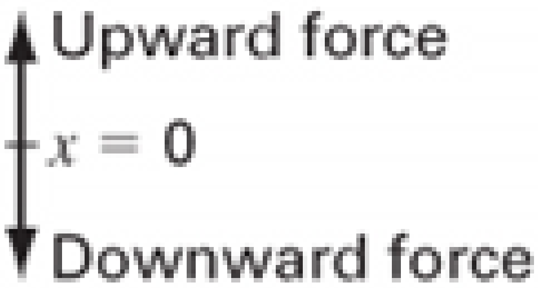


Conservation of momentum in civil engineering



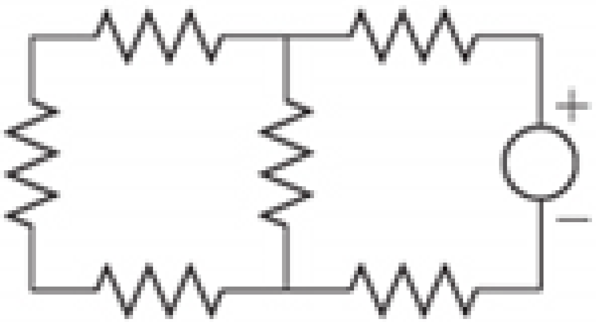
Conservation of momentum in mechanical engineering

Force balance:



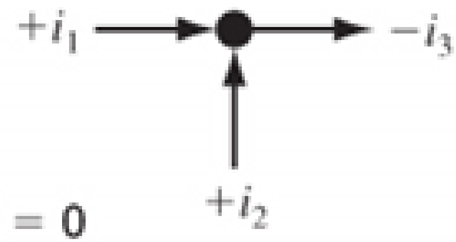
$$m\frac{d^2x}{dt^2} = \text{downward force} - \text{upward force}$$

Conservation of charge and energy in electrical engineering



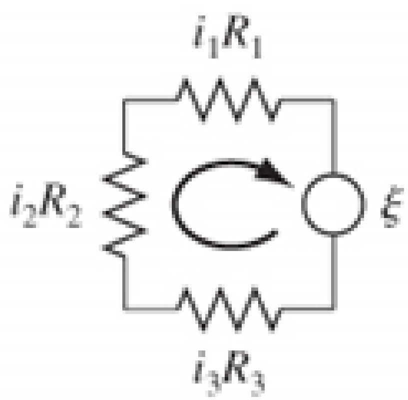
Circuit

Current balance:



For each node
Σ current (i) = 0

Voltage balance:



Around each loop
Σ emf's − Σ voltage drops for resistors = 0

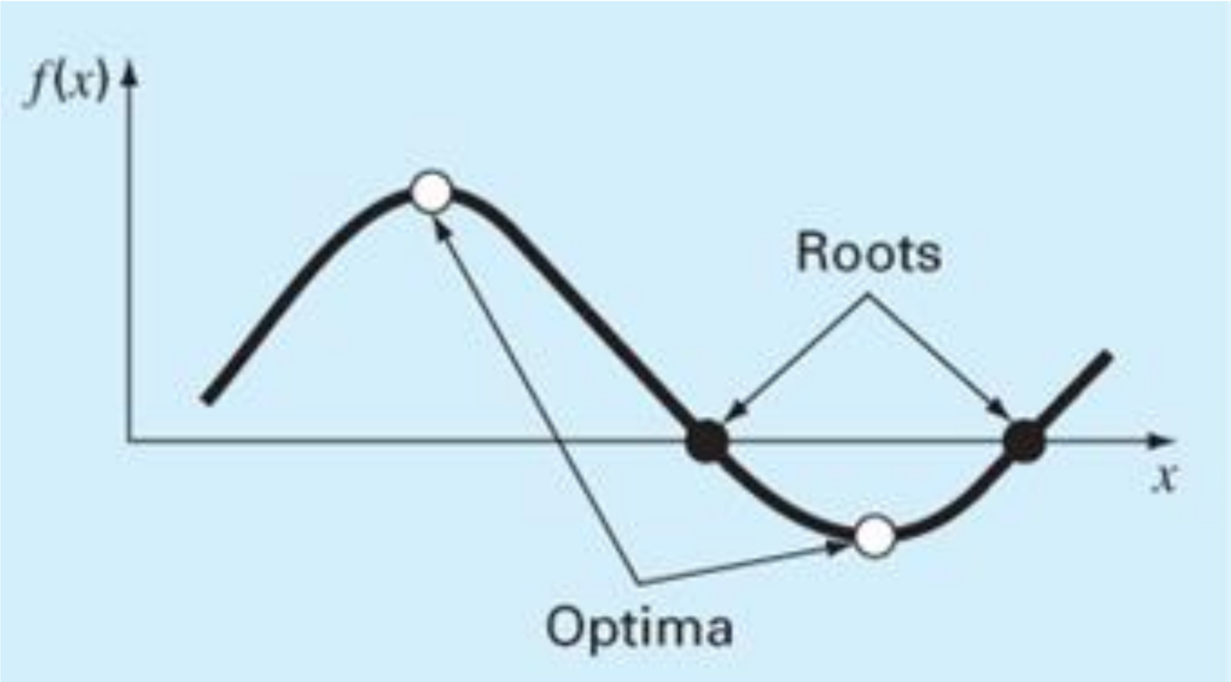
$\Sigma\,\xi - \Sigma\, iR = 0$

Major issues in this course

- **Roots and optimization**

  Roots: Solve for x so that f(x)=0

  Optimization: Solve for x so that f'(x)=0



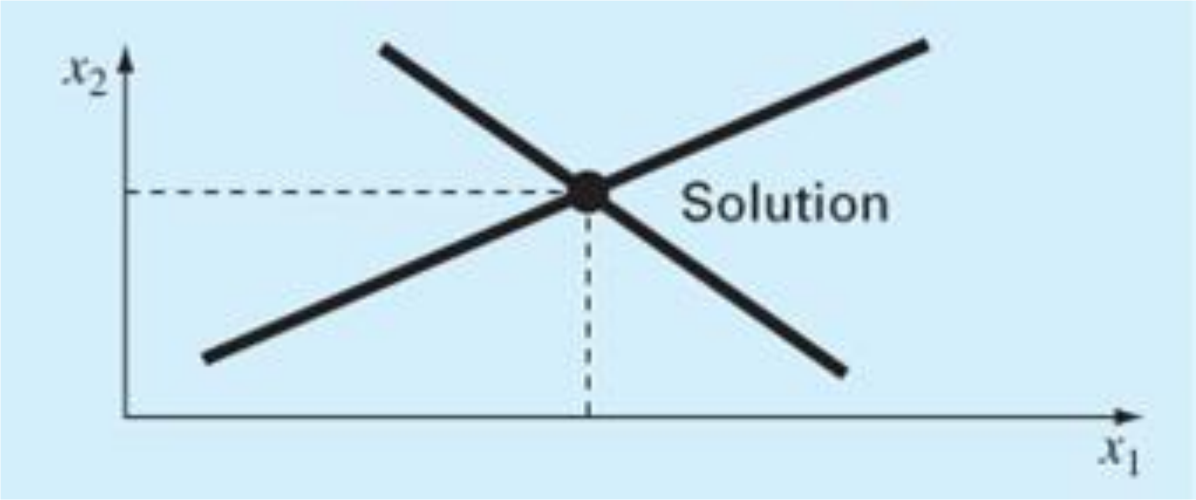
Major issues in this course (cont.)

- **Linear algebra equations**

  Solve for x's

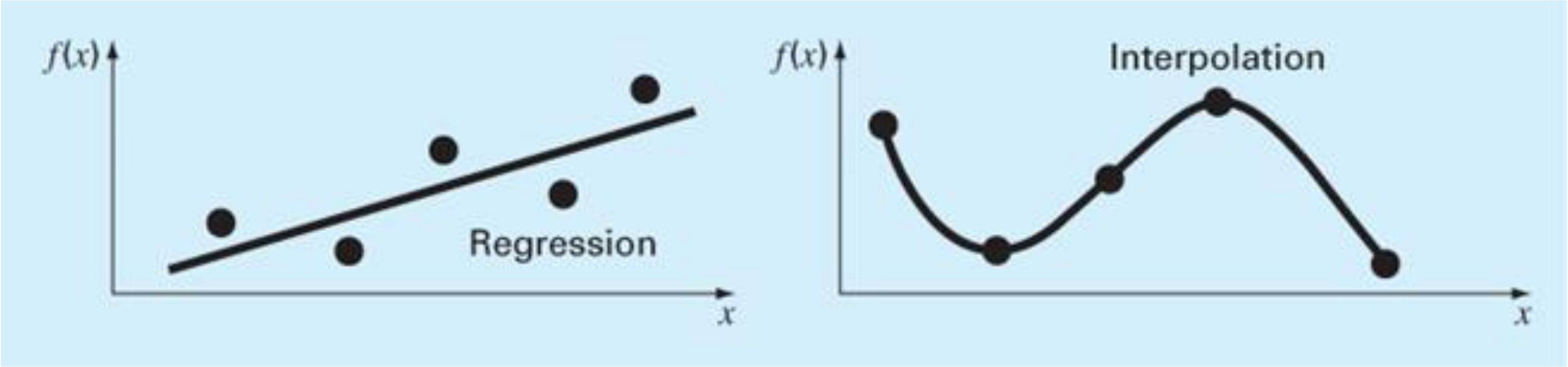
  $a_{11}x_1 + a_{12}x_2 = b_1$

  $a_{21}x_2 + a_{22}x_2 = b_2$



Major issues in this course (cont.)

- **Curve fitting**

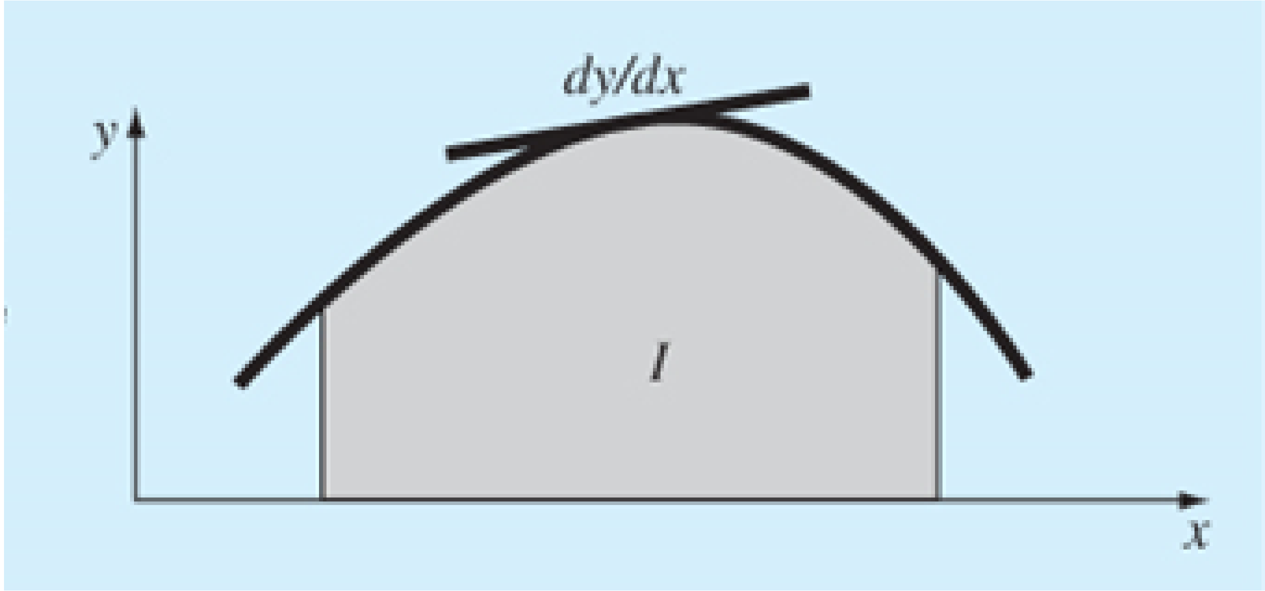


Major issues in this course (cont.)

- **Integration and differentiation**

  Integration: Find area under curve

  Differentiation: Find slope of curve



Grading policy

- Midterm & final examinations 75%
- Classworks + attendance + efforts + Homeworks 25%

Reference

- Textbook
  - Steven C. Chapra "Applied Numerical Methods with MATLAB" , 3rd ed., McGraw Hill, 2012.
- Reference
  - Steven C. Chapra, Raymond Canale “Numerical Methods for Engineers, 7th ed., McGraw Hill, 2015.
  - T. Sauer "Numerical Analysis", 2nd ed., Pearson, 2014.