Eigenvalues

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Mathematical background of eigenvalues

- Linear nonhomogenous system

  $[A]\ \{x\} = \{b\}$

- Linear homogenous system

  $[A]\ \{x\} = \{0\}$

- Eigenvalue problems

  $[[A] - \lambda\ [I]]\ \{x\} = 0$

$$
\begin{aligned}
(a_{11} - \lambda)\ x_1 + \quad a_{12}\ x_2 + \quad a_{13}\ x_3 &= 0 \\
a_{21}\ x_1 + (a_{22} - \lambda)\ x_2 + \quad a_{23}\ x_3 &= 0 \\
a_{31}\ x_1 + \quad a_{32}\ x_2 + (a_{33} - \lambda)\ x_3 &= 0
\end{aligned}
$$

Two-equation case

$$\begin{aligned}(a_{11}-\lambda)x_1 + \qquad a_{12}x_2 &= 0\\ a_{21}x_1 + (a_{22}-\lambda)x_2 &= 0\end{aligned}$$

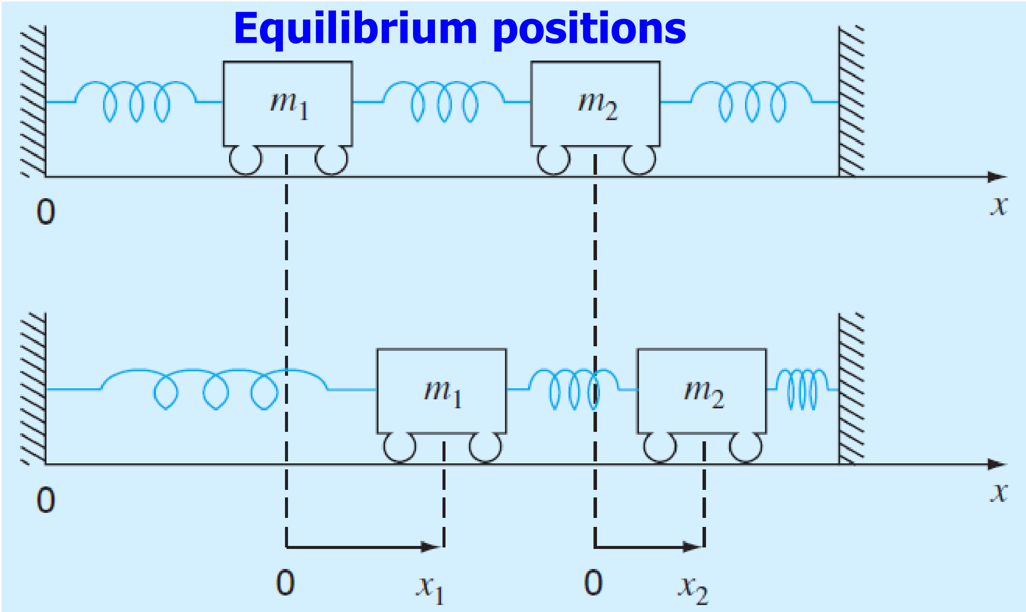
Characteristic polynomial

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2 - (a_{11}+a_{22})\lambda - a_{12}a_{21}$$

$$\begin{matrix}\lambda_1\\ \lambda_2\end{matrix} = \frac{(a_{11}-a_{22})^2 \pm \sqrt{(a_{11}-a_{22})^2 - 4a_{12}a_{21}}}{2}$$

Physical background of eigenvalues

- Oscillations or vibrations of mass-spring systems



$$m_1 \frac{d^2 x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k(x_2 - x_1) - kx_2$$

Mass-spring systems (cont.)

From vibration theory

$$x_i = X_i \sin(\omega t)$$

$$m_1 \frac{d^2 x_1}{dt^2} = -k x_1 + k(x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k(x_2 - x_1) - k x_2$$

$$\left(\frac{2k}{m_1} - \omega^2\right) X_1 - \frac{k}{m_1} X_2 = 0$$

$$-\frac{k}{m_2} X_1 + \left(\frac{2k}{m_2} - \omega^2\right) X_2 = 0$$

The solution is reduced to an eigenvalue problem:

$$\boldsymbol{\lambda = \omega^2}$$

Example of mass-spring systems

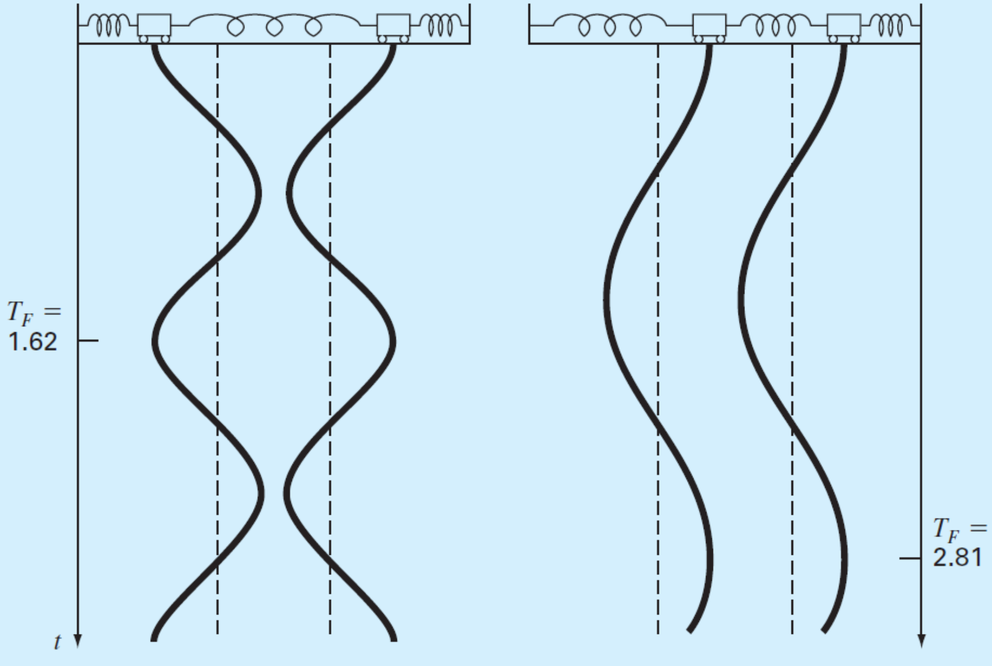
If $m_1 = m_2 = 40$ kg and $k = 200$ N/m

$$(10 - \lambda)x_1 - 5x_2 = 0$$
$$-5x_1 + (10 - \lambda)x_2 = 0$$

$\lambda = \omega^2 = 15, 5$

$\omega = 3.873, 2.36$ radian/s

$f = 0.6164, 0.3559$ Hz



(*a*) First mode (*b*) Second mode

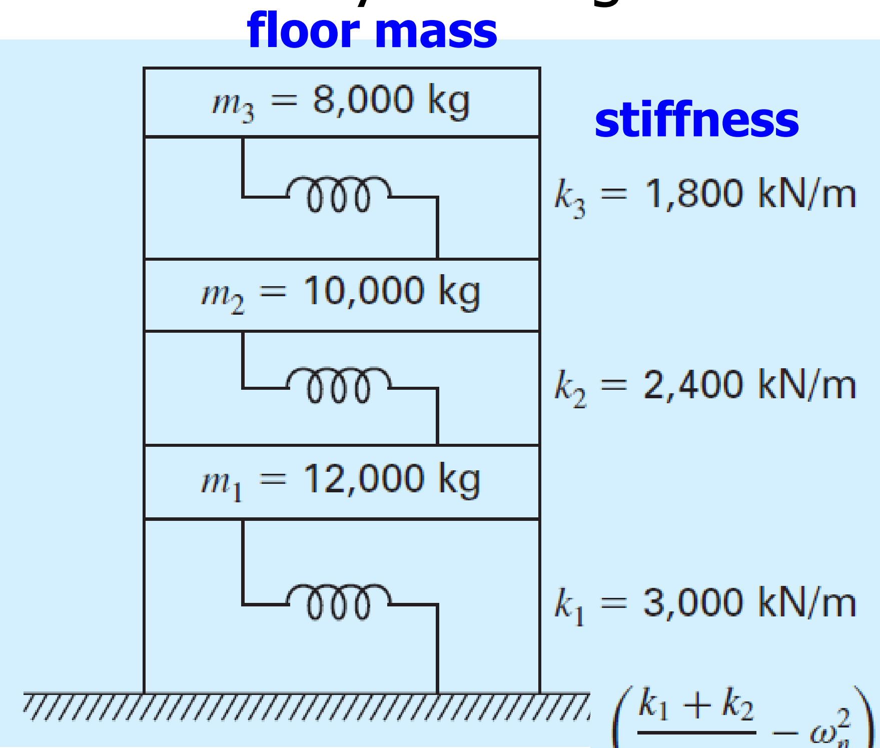
Determining eigenvalues & eigenvectors with MATLAB

```
A = [10 -5;-5 10];
[v,lambda] = eig(A)
v =
    -0.7071    -0.7071
    -0.7071     0.7071
lambda =
     5      0
     0     15
```

eigenvectors

Dynamics of building under the influence of earthquakes

- A three-story building modeled as a mass-spring system



$$\left(\frac{k_1 + k_2}{m_1} - \omega_n^2\right) X_1 - \frac{k_2}{m_1} X_2 = 0$$

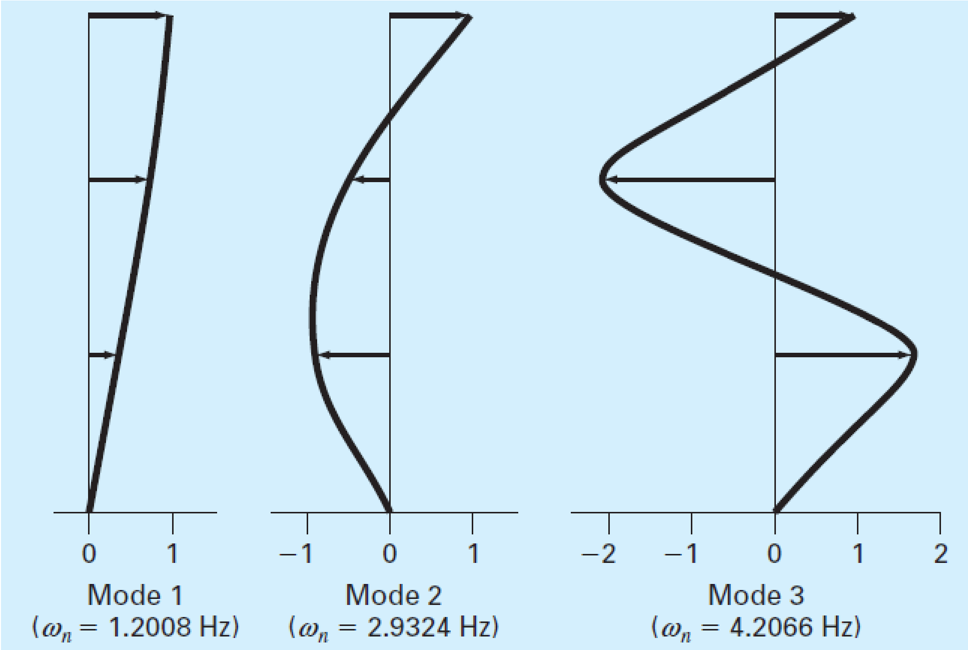
$$-\frac{k_2}{m_2} X_1 + \left(\frac{k_2 + k_3}{m_2} - \omega_n^2\right) X_2 - \frac{k_3}{m_2} X_3 = 0$$

$$-\frac{k_3}{m_3} X_2 + \left(\frac{k_3}{m_3} - \omega_n^2\right) X_3 = 0$$

Dynamics of three-floor building (cont.)

$$
\begin{aligned}
(450-\omega_n^2)X_1 \quad - 200X_2 \quad &= 0 \\
-240X_1 + (420-\omega_n^2)X_2 \quad - 180X_3 &= 0 \\
-225X_2 + (225-\omega_n^2)X_3 &= 0
\end{aligned}
$$

```
A=[450 -200 0;-240 420 -180;0 -225 225];
[v,d]=eig(A);
wn=sqrt(diag(d))/2/pi;
```

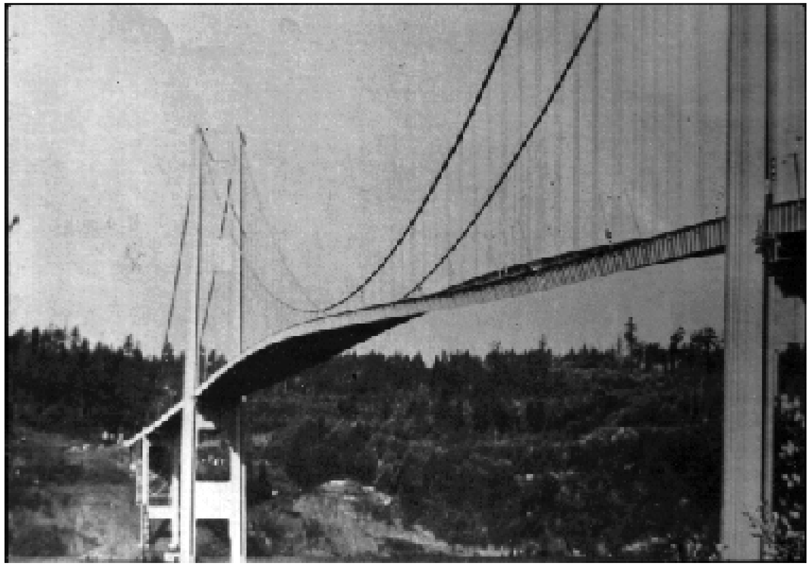


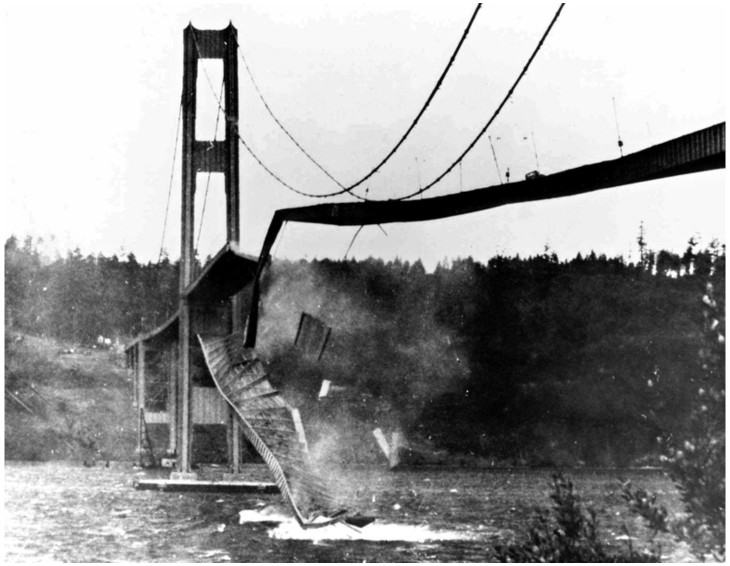
Dynamics of three-floor building (cont.)

- Natural frequencies are characteristics of floor's structure
- The frequency content of an earthquake typically has the most energy between 0 to 20 Hz, a spectrum of frequencies with varying amplitudes.
- When these amplitudes coincide with the natural frequencies of buildings, large dynamic responses are induced, creating large stresses and strains in the structure.

Tacoma narrows bridge collapse in 1940

- An example of elementary forced resonance, with the wind providing an external periodic frequency that matched the bridge's natural structural frequency.





https://www.youtube.com/watch?v=nFzu6CNtqec

Reference

- Steven C. Chapra "Applied Numerical Methods with MATLAB", 3rd ed., McGraw Hill, 2012.